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Analytical Approximate Solution of Coupled Wave Equations with a Nonlinear Stiffness

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14. ABSTRACT

By utilizing the Euler-Lagrange equations, a set of coupled partial differential equations was derived for two crossing strings with a spring at the crossover point. This report considers three types of springs: a nonlinear softening spring, a nonlinear hardening spring, and a linear spring. The Adomian decomposition method was used to obtain an analytical approximate solution from the derived coupled wave equations. The dynamic responses of the analytical solutions for both the nonlinear softening and nonlinear hardening springs were compared to the response from the linear spring. In both the nonlinear softening and nonlinear hardening springs, higher frequency oscillations were observed at the spring location; however, when a linear spring was used, the strings did not exhibit higher frequency oscillations and only oscillated in the mode by which they were excited.

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ANALYTICAL APPROXIMATE SOLUTION OF COUPLED WAVE EQUATIONS WITH A NONLINEAR STIFFNESS

INTRODUCTION

PURPOSE

Equations of motion can be derived for complex dynamical systems by using calculus of variations; however, because of the nonlinear complexities of these equations, closed-form solutions are usually unattainable. An alternative approach is to interrogate the derived equations by numerical methods. Although the numerical methods do allow the capture of the dynamic response of the system, a closed-form solution is always more desirable. This report describes the Naval Undersea Warfare Center (NUWC) Division Newport's effort to develop a method to derive an approximate closed-form solution of the coupled wave equation.

BACKGROUND

In the early 1990s, Adomian developed a method to derive analytical approximate solutions to nonlinear functional equations.[1] This method is referred to as the "Adomian decomposition method" (ADM). The solution to the given nonlinear functional equation can be approximated by an infinite series solution of the linear and nonlinear terms, provided the nonlinear terms are represented by a sum of series of Adomian polynomials.[2, 3, 4] ADM has been successfully applied to various types of ordinary,[5, 6, 7] partial,[8, 9, 10, 11, 12] and delay differential equations[13] to develop closed-form approximate solutions.

SCOPE

In this report, the derivation of an approximate closed-form solution of the coupled wave equation is accomplished by using the ADM. The coupling is realized by both the nonlinear softening and the nonlinear hardening springs and a linear spring located at the center of the crossing strings. The dynamic response of the closed-form solutions is compared for each spring type under the same prescribed initial condition.

The remainder of this report explores the Adomian decomposition method and provides a brief derivation of the coupled wave equations; derives approximate analytical solutions for the coupled wave equation using the Adomian decomposition method; and investigates the closed-form solutions for the three different types of springs under the same prescribed initial condition.

ADOMIAN DECOMPOSITION METHOD

GENERAL

Consider equation (1),

$$Fu = g(t)\,, \tag{1}$$

where F is a nonlinear operator, and, when it is expanded into linear and nonlinear terms, results in,

$$Lu + Ru + Nu = g(t). \tag{2}$$

The linear term is represented by Lu, L is invertible and is chosen to be the highest order derivative, R is the remainder of the linear terms, $g(t)$ is the nonhomogeneous right side, and the nonlinear terms are represented by Nu. If L is chosen to be a derivative of order n, then L^{-1} will be an n-fold integral. For example, if

$$L - \frac{d^2}{dt^2}[\cdot] \text{ then } L^{-1} = \int_0^t \int_0^t [\cdot] dtdt,$$

and

$$L^{-1}L = u(0) + tu'(0).$$

Solving equation (2) for Lu results in

$$Lu = -Ru - Nu + g(t). \tag{3}$$

Adomian postulated that the solution u can be approximated by an infinite sum of series

$$u = \sum_{n=0}^{\infty} u_n\,, \tag{4}$$

and the nonlinear term Nu can be approximated by a sum of Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n)\,, \tag{5}$$

where the Adomian polynomials of $u_0, u_1, ..., u_n$ can be calculated by the formula

$$A_n(u_0,u_1,...,u_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}u_i\lambda^i\right)\right]_{\lambda=0}, \tag{6}$$

where λ is a dummy variable. The first three Adomian polynomials are shown for convenience,

$$\begin{aligned} A_0 &= N(u_0) \\ A_1 &= u_1 N'(u_0) \\ A_2 &= \frac{u_1^2}{2!}N'(u_0)+u_2N(u_0), \end{aligned} \tag{7}$$

where $(\cdot)'$ denotes the first derivative with respect to u. Substituting equation (4) and equation (5) into equation (3) results in

$$L\sum_{n=0}^{\infty}u_n=-R\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}A_n(u_0,u_1,...,u_n)+g(t), \tag{8}$$

and operating on equation (8) with its inverse L^{-1} yields

$$\sum_{n=0}^{\infty}u_n=u(0)+tu'(0)-L^{-1}R\sum_{n=0}^{\infty}u_n-L^{-1}\sum_{n=0}^{\infty}A_n(u_0,u_1,...,u_n)+L^{-1}g(t). \tag{9}$$

Finally, equation (9) can be rewritten in a recursive relation as in equation (10),

$$\begin{aligned} u_0 &= u(0)+tu'(0)+L^{-1}g(t), \\ u_n &= -L^{-1}Ru_{n-1}-L^{-1}A_{n-1},\ n\geq 1, \end{aligned} \tag{10}$$

where u_0 is obtained from the prescribed initial and boundary conditions. The *k*-term approximation can be used as a series solution,

$$\phi_k = u \approx \sum_{i=0}^{k-1}u_i\,, \tag{11}$$

where the exact solution is given by

$$u=\lim_{n\to\infty}u_n. \tag{12}$$

If more accuracy is desirable, then more terms should be included in the series solution.

WAVE EQUATION DERIVATION

Using the Euler-Lagrange equations allows the coupled wave equations to be derived where the coupling is realized through a nonlinear softening spring. Consider the two crossing strings depicted in figure 1, where the strings lie in the x_1x_2-plane. In this report, the strings are considered to be orthogonal to each other throughout the simulation. Shearing or twisting effects of the strings are not considered. The string in the x_1-direction has a vertical displacement $u(x_1, x_2, t)$, and the string in the x_2-plane has a vertical displacement $v(x_1, x_2, t)$.

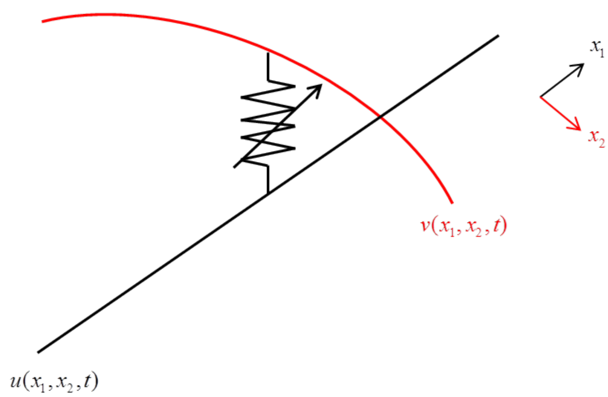

Figure 1. Two Crossing Strings Where the Coupling Is Realized with a Nonlinear Softening Spring

The Lagrangian can be defined as

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) \equiv T - V_f - V_s,$$

where T is the kinetic energy, V_f is the potential energy due to the work required to displace a differential element of the string, and V_s is the potential energy of the nonlinear spring where a nominal spring length is used. After substituting in the terms, the Lagrangian can be expressed as

$$\begin{aligned} L = {} & \frac{1}{2}\rho\left(\left(\frac{\partial}{\partial t}u(x_1,x_2,t)\right)^2 + \left(\frac{\partial}{\partial t}v(x_1,x_2,t)\right)^2\right) \\ & -\frac{E}{2}\left(\left(\frac{\partial u(x_1,x_2,t)}{\partial x_1} + \frac{\partial u(x_1,x_2,t)}{\partial x_2}\right)^2 + \left(\frac{\partial v(x_1,x_2,t)}{\partial x_1} + \frac{\partial v(x_1,x_2,t)}{\partial x_2}\right)^2\right) \\ & -\left(\frac{K_L}{2}\left(v(x_1,x_2,t) - u(x_1,x_2,t)\right)^2 - \frac{K_{NL}}{4}\left(v(x_1,x_2,t) - u(x_1,x_2,t)\right)^4\right), \end{aligned} \tag{13}$$

where ρ is the mass density of the string, E is the Young's modulus of the string, and K_L and K_{NL} are the linear and nonlinear stiffness coefficients, respectively. In this report, ρ and E are kept the same in both strings. In its normal form, the Euler-Lagrangian differential equation is expressed as

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \tag{14}$$

where q is the generalized coordinate and $\dot{q}$ is the time derivative. The system of interest here, however, is continuous, so the generalized coordinates are $u(x_1, x_2, t)$ and $v(x_1, x_2, t)$; thus, the Euler-Lagrangian differential equations for $u(x_1, x_2, t)$ and $v(x_1, x_2, t)$ are

$$\begin{aligned}
&\frac{\partial L}{\partial u(x_1,x_2,t)} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial\left(\frac{\partial u(x_1,x_2,t)}{\partial t}\right)}\right] - \frac{\partial}{\partial x_1}\left[\frac{\partial L}{\partial\left(\frac{\partial u(x_1,x_2,t)}{\partial x_1}\right)}\right] - \frac{\partial}{\partial x_2}\left[\frac{\partial L}{\partial\left(\frac{\partial u(x_1,x_2,t)}{\partial x_2}\right)}\right] = 0 \\
&\frac{\partial L}{\partial v(x_1,x_2,t)} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial\left(\frac{\partial v(x_1,x_2,t)}{\partial t}\right)}\right] - \frac{\partial}{\partial x_1}\left[\frac{\partial L}{\partial\left(\frac{\partial v(x_1,x_2,t)}{\partial x_1}\right)}\right] - \frac{\partial}{\partial x_2}\left[\frac{\partial L}{\partial\left(\frac{\partial v(x_1,x_2,t)}{\partial x_2}\right)}\right] = 0.
\end{aligned} \tag{15}$$

Evaluating the terms yields the coupled wave equations

$$\begin{aligned}
\frac{\partial^2 u(x_1,x_2,t)}{\partial t^2} &= c_1^2 \nabla^2 u(x_1,x_2,t) + K_L\left(v(x_1,x_2,t) - u(x_1,x_2,t)\right) - K_{NL}\left(v(x_1,x_2,t) - u(x_1,x_2,t)\right)^3 \\
\frac{\partial^2 v(x_1,x_2,t)}{\partial t^2} &= c_2^2 \nabla^2 v(x_1,x_2,t) + K_L\left(u(x_1,x_2,t) - v(x_1,x_2,t)\right) - K_{NL}\left(u(x_1,x_2,t) - v(x_1,x_2,t)\right)^3,
\end{aligned} \tag{16}$$

where

$$c_1^2 = \sqrt{\frac{E}{\rho}} \text{ and } c_2^2 = \sqrt{\frac{E}{\rho}}.$$

ADM APPLIED TO THE COUPLED WAVE EQUATION

According to the ADM, the solution to $u = u(x_1, x_2, t)$ and $v = v(x_1, x_2, t)$ can be represented as

$$u(x_1, x_2, t) = \sum_{n=0}^{\infty} u_n(x_1, x_2, t)$$
and
$$v(x_1, x_2, t) = \sum_{n=0}^{\infty} v_n(x_1, x_2, t). \tag{17}$$

Because the summation of u_n and v_n is the desired solution and converges rapidly, the k -terms of the summation can be calculated. The resulting approximate analytical solution, therefore, is

$$\phi_k = \sum_{n=0}^{k-1} u_n(x_1, x_2, t)$$
and
$$\varphi_k = \sum_{n=0}^{k-1} v_n(x_1, x_2, t). \tag{18}$$

Because the nonlinear terms are functions of two variables, one cannot continue with the Adomian polynomials described in equation (6). The Adomian polynomials for the two displacement variables $u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n$ are described by

$$B_n(u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} u_i \lambda^i, \sum_{i=0}^{\infty} v_i \lambda^i\right)\right]_{\lambda=0}, \tag{19}$$

where the first three terms are shown for convenience:

$$\begin{aligned}
B_0 &= N(u_0, v_0)\\
B_1 &= u_1 \frac{\partial}{\partial u} N(u_0, v_0) + v_1 \frac{\partial}{\partial v} N(u_0, v_0)\\
B_2 &= \frac{u_1^2}{2!}\frac{\partial^2}{\partial u^2} N(u_0, v_0) + \frac{v_1^2}{2!}\frac{\partial^2}{\partial v^2} N(u_0, v_0) + u_1 v_1 \frac{\partial^2}{\partial u \partial v} N(u_0, v_0) + u_2 \frac{\partial}{\partial u} N(u_0, v_0) + v_2 \frac{\partial}{\partial v} N(u_0, v_0).
\end{aligned} \tag{20}$$

For the remainder of this report, $u = u(x_1, x_2, t)$ and $v = v(x_1, x_2, t)$. From the coupled wave equations described in equation (16), the highest-order derivative is second order, so the linear operator becomes $L = \frac{\partial^2}{\partial t^2}$ and $L^{-1} = \int_0^t \int_0^t (\cdot) dt dt$. In the u equation, the nonlinear term is $N(u, v) = K_{NL}(v - u)^3$ and $N(u, v) = K_{NL}(u - v)^3$ for the v equation. The first three terms of the

Adomian polynomial series expansion are shown for convenience where the superscripts u, v denote the polynomials for the u, v equations, respectively:

$$
\begin{aligned}
B_0^u &= K_{NL}(v_0 - u_0)^3 \\
B_1^u &= 3K_{NL}\left[(v_0 - u_0)^2(v_1 - u_1)\right] \\
B_2^u &= K_{NL}\left[3(v_1 - u_1)^2(v_0 - u_0) + \frac{3}{2}(v_0 - u_0)^2(2v_2 - 2u_2)\right] \\
B_3^u &= K_{NL}\left[(v_1 - u_1)^3 + 3(v_0 - u_0)(v_1 - u_1)(2v_2 - 2u_2) + \frac{1}{2}(v_0 - u_0)^2(6v_3 - 6u_3)\right] \\
B_0^v &= -B_0^u \\
B_1^v &= -B_1^u \\
B_2^v &= -B_2^u \\
B_3^v &= -B_3^u.
\end{aligned}
\tag{21}
$$

The n^{th} recursive equation for both u and v are found by plugging equation (16) into equation (10), which yields:

$$
\begin{aligned}
u_n &= \int_0^t\int_0^t \left[c_1^2\nabla^2 u_{n-1} + K_L\left(v_{n-1} - u_{n-1}\right)\right]dtdt - \int_0^t\int_0^t B_{n-1}^u(u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1})dtdt \\
v_n &= \int_0^t\int_0^t \left[c_2^2\nabla^2 v_{n-1} + K_L\left(u_{n-1} - v_{n-1}\right)\right]dtdt - \int_0^t\int_0^t B_{n-1}^v(u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1})dtdt.
\end{aligned}
\tag{22}
$$

Because the solution is a sum of iterates, it must be transformed into a closed-form solution. This transformation is achieved by taking a Laplace transform of both ϕ_k and φ_k with respect to t. A Pade approximation of order [2, 2] was used to approximate the coefficients of the Laplace transformed polynomials into a rational function.[14] An inverse Laplace transform was used to convert back to the time domain to yield a closed-form solution.

SIMULATION RESULTS

In this section, the approximate solutions are computed for three types of springs with the same initial condition with the following system parameters over the domain Θ. Both strings experience an initial displacement (u_0 and v_0) as shown in equation (23) with zero initial velocity (u_1 and v_1):

$$u_0 = \sin(\pi x_1)\sin(\pi x_2),\ v_0 = -\frac{1}{2}\sin(\pi x_1)\sin(\pi x_2),\ u_1 = v_1 = 0$$
$$c_1^2 = c_2^2 = 1,\ g(t)=0,\ \Theta = [0,1]\times[0,1]. \tag{23}$$

The following spring constants were chosen:

(1) nonlinear softening spring $K_L = 1,\ K_{NL} = -3$
(2) nonlinear hardening spring $K_L = 1,\ K_{NL} = 3$ (24)
(3) linear spring $K_L = 1,\ K_{NL} = 0.$

In the first case, the steps used toward deriving the closed-form solutions are presented; however, in the other two cases, only the responses are provided.

CASE 1

In this case, a nonlinear softening spring is used. Only the first three iterations are shown due to the number of terms; four iterations, however, were used to develop the closed-form solution

$$
\begin{aligned}
u_0 &= \sin(\pi x_1)\sin(\pi x_2) \\
v_0 &= -\frac{1}{2}\sin(\pi x_1)\sin(\pi x_2) \\
u_1 &= -\frac{t^2}{16}\sin(\pi x_1)\sin(\pi x_2)\left(16\pi^2 + 93 - 81\cos^2(\pi x_2) - 81\cos^2(\pi x_1) + 81\cos^2(\pi x_1)\cos^2(\pi x_2)\right) \\
v_1 &= t^2\sin(\pi x_1)\sin(\pi x_2)\left(8\pi^2 + 93 - 81\cos^2(\pi x_2) - 81\cos^2(\pi x_1) + 81\cos^2(\pi x_1)\cos^2(\pi x_2)\right) \\
u_2 &= \frac{t^4}{192}\sin(\pi x_1)\sin(\pi x_2)\begin{pmatrix} 22785 + 32\pi^4 - 3888\pi^2(\cos^2(\pi x_1) + \cos^2(\pi x_2)) \\ +81810\cos^2(\pi x_1)\cos^2(\pi x_2) + 3450\pi^2 \\ -42444(\cos^2(\pi x_1) + \cos^2(\pi x_2)) + 4374\pi^2\cos^2(\pi x_1)\cos^2(\pi x_2) \\ -39366(\cos^4(\pi x_2)\cos^2(\pi x_1) + \cos^4(\pi x_1)\cos^2(\pi x_2)) \\ +19683(\cos^4(\pi x_1) + \cos^4(\pi x_2) + \cos^4(\pi x_1)\cos^4(\pi x_2)) \end{pmatrix} \\
v_2 &= -\frac{t^4}{192}\sin(\pi x_1)\sin(\pi x_2)\begin{pmatrix} 22785 + 32\pi^4 - 3888\pi^2(\cos^2(\pi x_1) + \cos^2(\pi x_2)) \\ +81810\cos^2(\pi x_1)\cos^2(\pi x_2) + 3450\pi^2 \\ -42444(\cos^2(\pi x_1) + \cos^2(\pi x_2)) + 4374\pi^2\cos^2(\pi x_1)\cos^2(\pi x_2) \\ -39366(\cos^4(\pi x_2)\cos^2(\pi x_1) + \cos^4(\pi x_1)\cos^2(\pi x_2)) \\ +19683(\cos^4(\pi x_1) + \cos^4(\pi x_2) + \cos^4(\pi x_1)\cos^4(\pi x_2)) \end{pmatrix}.
\end{aligned}
\tag{25}
$$

After four iterations, the approximate solution for u is $\phi_4(x_1, x_2, t) = \sum_{n=0}^{3} u_n$; for v, the approximate solution is $\varphi_4(x_1, x_2, t) = \sum_{n=0}^{3} u_n$. The Laplace transform of $\phi_4(x_1, x_2, t)$ and $\varphi_4(x_1, x_2, t)$ yield $\phi_4(x_1, x_2, s)$ and $\varphi_4(x_1, x_2, s)$. The Pade approximate of order [2, 2] for both $\phi_4(x_1, x_2, s)$ and $\varphi_4(x_1, x_2, s)$ is

$$
\begin{aligned}
\phi_4(x_1,x_2,s) &= \frac{81\sin(\pi x_1)\sin(\pi x_2)}{s\left(81+\dfrac{\frac{6561}{8}\left(\cos^2(\pi x_1)\cos^2(\pi x_2)-\cos^2(\pi x_1)-\cos^2(\pi x_2)\right)+162\pi^2+\frac{7533}{8}}{s^2}\right)} \\
\varphi_4(x_1,x_2,s) &= -\frac{9\sin(\pi x_1)\sin(\pi x_2)}{s\left(81+\dfrac{\frac{6561}{4}\left(\cos^2(\pi x_1)\cos^2(\pi x_2)-\cos^2(\pi x_1)-\cos^2(\pi x_2)\right)+162\pi^2+\frac{7533}{4}}{s^2}\right)}.
\end{aligned}
\tag{26}
$$

The final closed-form solution is:

$$
\begin{aligned}
u(x_1,x_2,t) &= \sin(\pi x_1)\sin(\pi x_2)\cosh\left(\frac{t}{4}\sqrt{-186-32\pi^2+162\cos^2(\pi x_1)-162\cos^2(\pi x_2)(\cos^2(\pi x_1)-1)}\right) \\
v(x_1,x_2,t) &= -\frac{1}{2}\sin(\pi x_1)\sin(\pi x_2)\cosh\left(\frac{t}{2}\sqrt{-93-8\pi^2+81\cos^2(\pi x_1)-81\cos^2(\pi x_2)(\cos^2(\pi x_1)-1)}\right).
\end{aligned}
\tag{27}
$$

The dynamic response to the prescribed initial condition is shown in figures 2 and 3. Figure 2 depicts the response of $v(x_1,x_2,t)$ (blue line) along the x_2-direction and $u(x_1,x_2,t)$ (red line) along the x_1-direction at time snaps t = 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0. Figure 3 shows a projection of these two responses for $v(x_1,x_2,t)$ (blue line) and $u(x_1,x_2,t)$ (red line) along the same coordinate axis to compare the oscillations between the two strings. Initially, the spring is compressed with the string along the x_1-direction having an initial displacement with a greater magnitude than the string along the x_2-direction. The effect of the nonlinear softening spring becomes evident as time progresses. The center location of both strings, where the spring is attached, oscillates at a higher frequency than the points closer to the fixed ends. In essence, this oscillation creates a rippling effect from the center of the string to the boundaries. If time were to progress further, the string would experience high-frequency oscillations throughout its length.

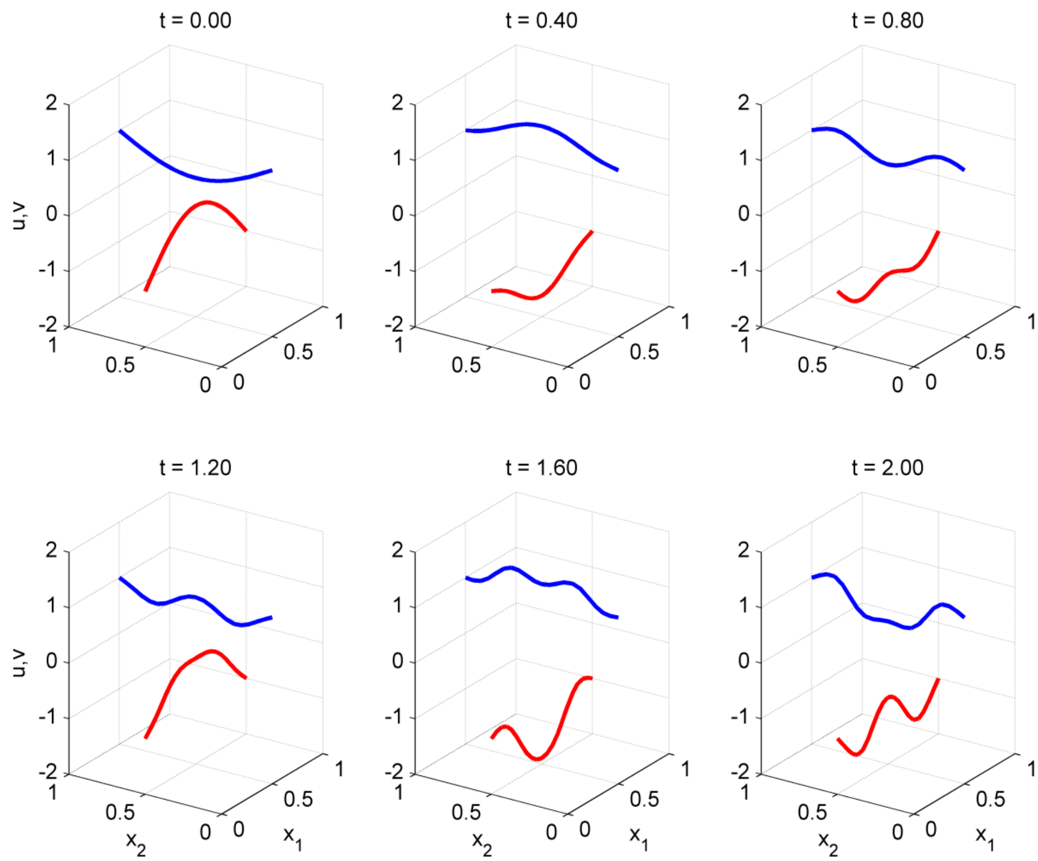

Figure 2. Response of $v(x_1, x_2, t)$ ***(blue line) Along the*** x_2***-Direction and*** $u(x_1, x_2, t)$ ***(red line) Along the*** x_1***-Direction for Case 1 at Time Snaps*** t ***= 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0***

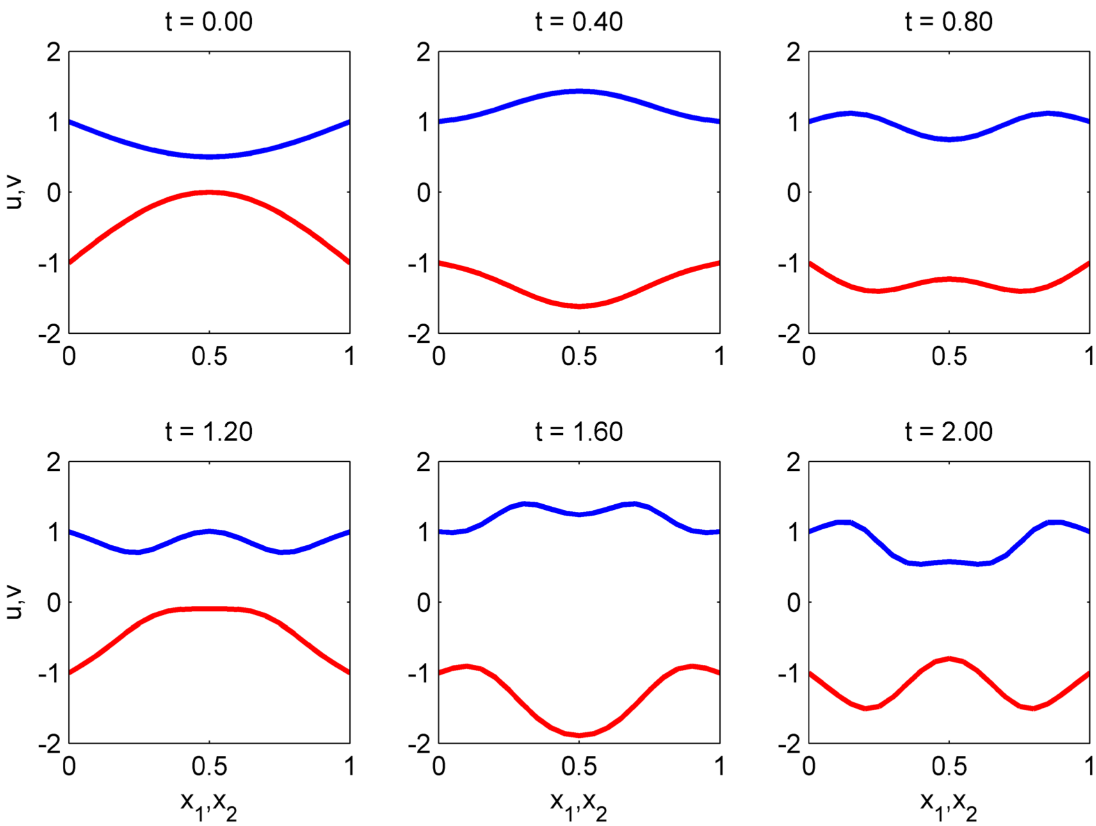

Figure 3. Projection View of the Response of $v(x_1, x_2, t)$ ***(blue line) and*** $u(x_1, x_2, t)$ ***(red line) Along the Same Axis for Case 1 at Time Snaps*** t ***= 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0***

CASE 2

In this case, a nonlinear hardening spring is used. The final closed-form solution is:

$$
\begin{aligned}
u(x_1,x_2,t) &= \sin(\pi x_1)\sin(\pi x_2)\cosh\left(t\sqrt{138-32\pi^2-162\cos^2(\pi x_1)+162\cos^2(\pi x_2)(\cos^2(\pi x_1)-1)}\right) \\
v(x_1,x_2,t) &= -\frac{1}{2}\sin(\pi x_1)\sin(\pi x_2)\cosh\left(t\sqrt{69-8\pi^2-81\cos^2(\pi x_1)+81\cos^2(\pi x_2)(\cos^2(\pi x_1)-1)}\right).
\end{aligned}
\tag{28}
$$

The dynamic responses depicted in figures 4 and 5 are similar to those for case 1 dynamics in figures 2 and 3. This similarity can be observed in the closed-form solutions in equations (28) and (27), which differ only by constants. The global oscillations remain the same, but the higher frequency oscillations are different from what was observed in the nonlinear softening spring. Figure 4 depicts the response of $v(x_1,x_2,t)$ (blue line) along the x_2-direction and the response of $u(x_1,x_2,t)$ (red line) along the x_1-direction at time snaps t = 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0; figure 5 shows a projection of these two responses for $v(x_1,x_2,t)$ (blue line) and $u(x_1,x_2,t)$ (red line) along the same coordinate axis. The effects of the nonlinear hardening characteristics of the spring are apparent in the response of the center location of the strings where the spring is located. Unlike the nonlinear softening case (case 1), both strings in case 2 have their inflection points at different times in the simulation.

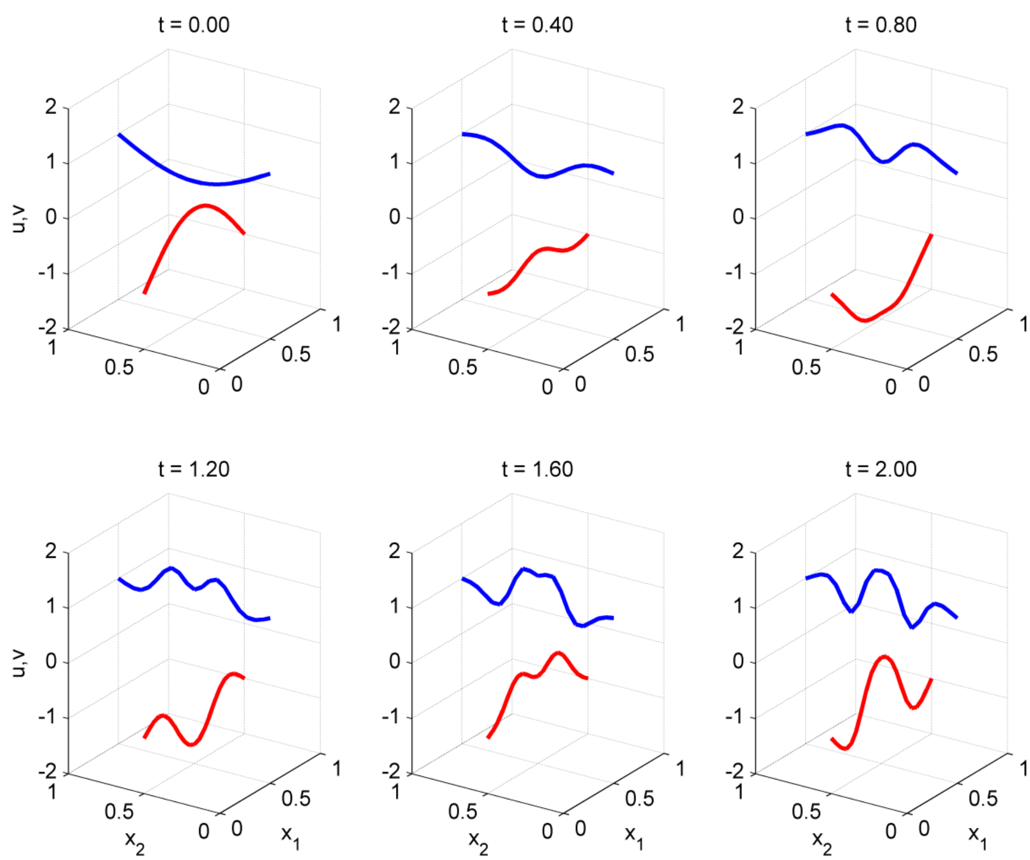

Figure 4. Response of $v(x_1,x_2,t)$ (blue line) Along the x_2-Direction and $u(x_1,x_2,t)$ (red line) Along the x_1-Direction for Case 2 at Time Snaps t = 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0

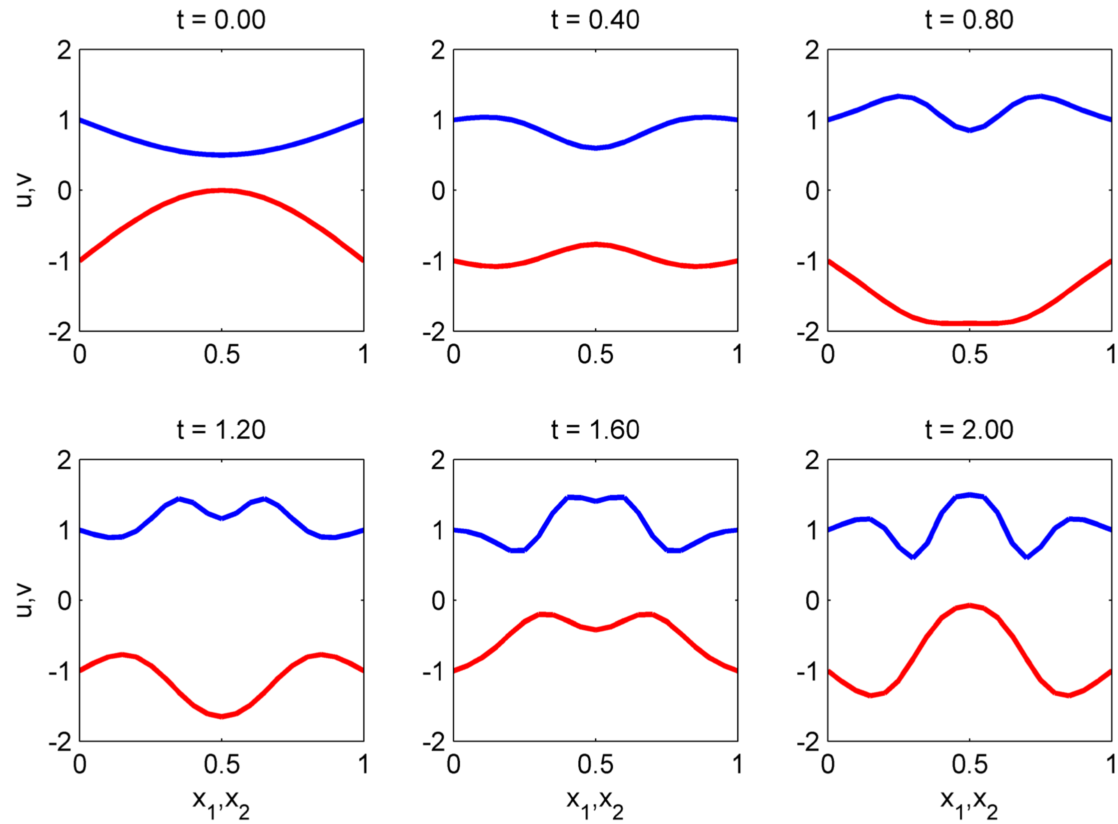

Figure 5. Projection View of the Response of $v(x_1, x_2, t)$ ***(blue line) and*** $u(x_1, x_2, t)$ ***(red line) Along the Same Axis for Case 2 at Time Snaps*** t ***= 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0***

CASE 3

In this case, a linear spring is used. The final closed-form solution is:

$$
\begin{aligned}
u(x_1,x_2,t) &= \sin(\pi x_1)\sin(\pi x_2)\cosh\left(\frac{t}{2}\sqrt{-8\pi^2-6}\right) \\
v(x_1,x_2,t) &= -\frac{1}{2}\sin(\pi x_1)\sin(\pi x_2)\cosh\left(t\sqrt{-2\pi^2-3}\right).
\end{aligned}
\tag{29}
$$

The dynamic response for case 3 is depicted in figures 6 and 7. Because the coupling spring connecting the two strings is linear and the initial condition is composed of just the first mode of oscillation for a string with fixed ends, both strings oscillate only in that first mode. The figures show that there are no higher frequency oscillations in the string, which can further be seen in the close-form solution in equation (29).

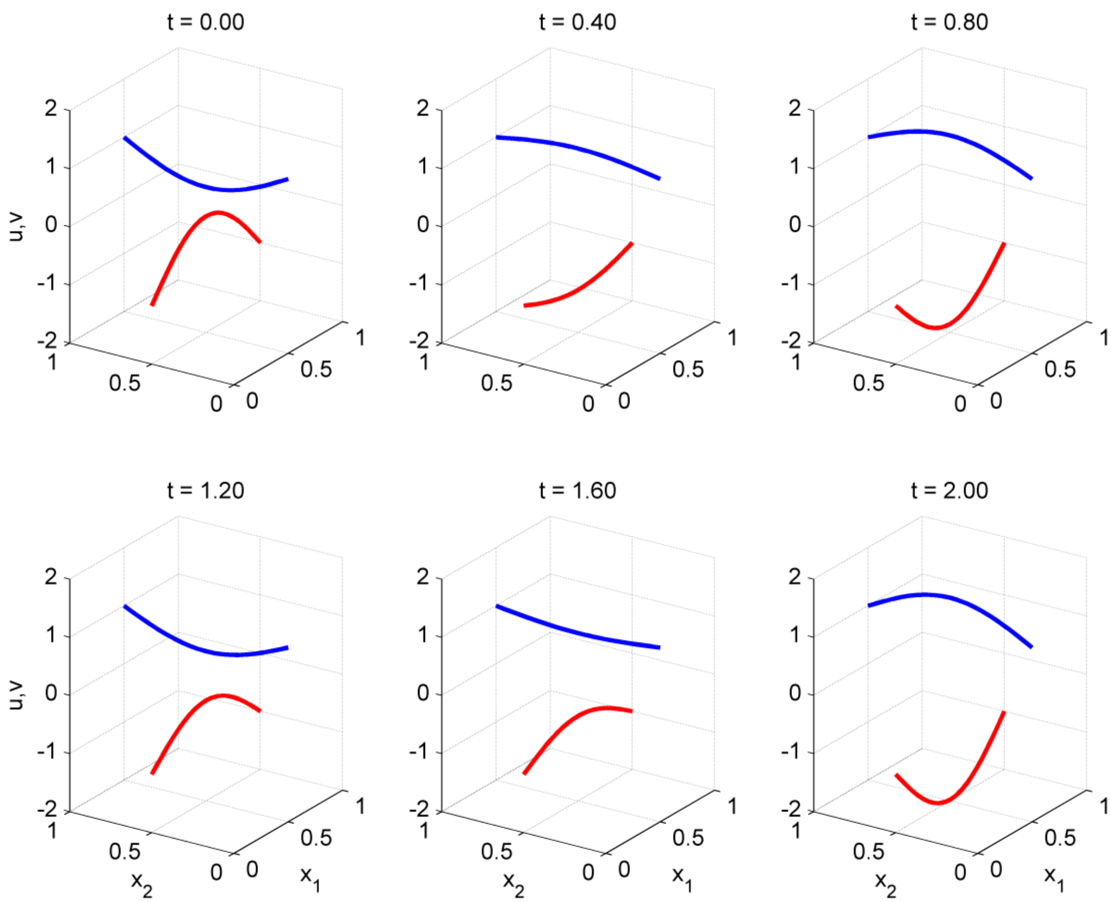

Figure 6. Response of $v(x_1,x_2,t)$ (blue line) Along the x_2-Direction and $u(x_1,x_2,t)$ (red line) Along the x_1-Direction for Case 3 at Time Snaps t = 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0

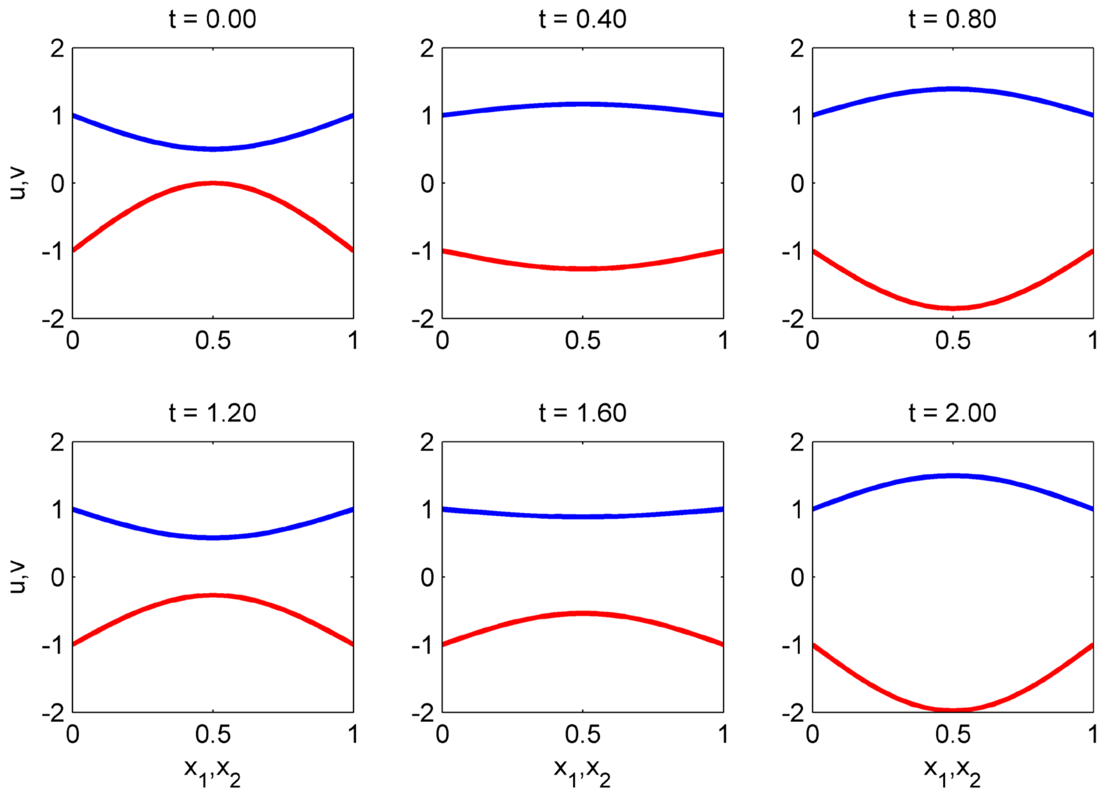

Figure 7. Projection View of the Response of** $v(x_1, x_2, t)$ **(blue line) and** $u(x_1, x_2, t)$ **(red line) Along the Same Axis for Case 3 at Time Snaps** t **= 0.0, 0.4, 0.8, 1.2, 1.6, and 2.0

CONCLUSIONS

The Adomian decomposition method was used to develop an approximate analytical solution for coupled wave equations. The coupling was realized by nonlinear softening and nonlinear hardening springs as well as for a linear spring. The equations for each type of spring were derived from the Euler-Lagrange equations. The effects of the nonlinear spring behaviors were evident in both the dynamic response and closed-form equations—very unlike the behavior observed for the linear spring case. In both the nonlinear softening and nonlinear hardening springs, higher frequency oscillations were observed at the spring location; however, when a linear spring was used, the strings exhibited no higher frequency oscillations and oscillated only in the mode by which they were excited.

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